FIRST-ORDER TRANSITIONS IN CHARGED BOSON NEBULAE

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Abstract

Using the first-order approximating solutions to the Einstein-Maxwell-Klein-Gordon system of equations for a complex scalar field minimally coupled to a spherically symmetric spacetime, we study the feedback of gravity and electric field on the charged scalar source. Within a perturbative approach, we compute, in the radiation zone, the transition amplitudes and the coherent source-field regeneration rate.

Keywords:

- Klein-Gordon-Maxwell-Einstein equations;
- charged boson stars.

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Boson stars, generally seen as gravitationally bound, both globally $U(1)$ and spherically symmetric, compact equilibrium configurations of cold complex scalar fields, were discovered theoretically, more than 30 years ago, by Kaup [1] and Ruffini and Bonazzola [2]. Later, a major interest has been focused on macroscopic stable boson stars, once they arose as promising candidates for non-baryonic dark matter in the universe [3]. Moreover, according to current observations, it has been suggested that a supermassive boson or soliton star could be at the center of our galaxy [4]. These assumptions raise the question whether such configurations, if they exist, are dynamically stable [5] and the settling down of a boson star to a stable state has been investigated via numerical calculations [6].

In our approach, we term by a boson star the charged scalar nebula which finds itself in one of the spherically symmetric positive-frequency modes of radial wave number k, with $k^2 \to 0_+$, and pulsation $\omega_k = [m_0^2 + k^2]^{1/2}$. Such a configuration is obviously unstable and the instabilities is expected to lead to the formation of boson star from a initially smooth state [7]. Not very much is specifically known in this direction [8], although the reversed stability to instability passage has been extensively investigated [5, 6, 9].

In what it concerns an analytical approach, exactly solvable models for boson stars with large selfinteraction [10] and for boson-fermion stars [11] have been worked out only in low dimensional gravity. In four dimensions, the bosonic or the mixed fermion-bosonic fields interacting via gravity have been investigated mainly by numerical calculations [1-6,8,9,12]. Therefore, the analytical solutions of the coupled field equations could be of interest for a better understanding of different stellar configurations as well as for a numerical-functional combined iterative treatment which describes the dynamics of charged boson stars formation.

In a previous letter [7], we have studied the $SO(3,1) \times U(1)$-gauged minimally coupled charged spinless field to a spherically symmetric spacetime and analytically obtained the first-order approximating solutions to the system

of Klein-Gordon-Maxwell-Einstein equations. These allow us to go further and to study now the feedback of gravity and electric field on the charged scalar source, computing perturbatively, in the long range approximation, the correspondingly induced transition amplitudes.

Let us consider the spherically symmetric configuration described by the metric

$$ds^2 \,=\, e^{2f}\,(dr)^2 \,+\, r^2\,\left(d\theta^2 + \sin^2\theta\,d\varphi^2\right) - \,e^{2h}\,(dt)^2\,, \tag{1}$$

where f and h are functions of r and t, and define the pseudo-orthonormal tetradic frame $\{e_a\}_{a=\overline{1,4}}$ with the corresponding dual orthonormal base

$$\omega^1 \,=\, e^f\,dr\;,\;\; \omega^2 \,=\, r\,d\theta\;,\;\; \omega^3 \,=\, r\sin\theta\,d\varphi\;,\;\; \omega^4 \,=\, e^h\,dt \tag{2}$$

The $SO(3,1)\times U(1)$-gauge invariant Lagrangean density for a charged boson of mass m_0, coupled to the electromagnetic field,

$$\mathcal{L} \,=\, \eta^{ab}\,\bar{\phi}_{;a}\phi_{;b} \,+\, m_0^2\,\bar{\phi}\phi \,+\, \frac{1}{4}\,F^{ab}F_{ab}\,, \tag{3}$$

lead to the following system of Klein-Gordon-Maxwell-Einstein equations

$$\Box\phi \,-\, m_0^2\,\phi \,=\, 2ie\,A^c\phi_{|c} \,+\, e^2A^cA_c\phi\;, \qquad \text{and its h.c.}\,, \tag{4}$$

$$F^{ab}{}_{:b} \,=\, -\,ie\,\eta^{ab}\left[\bar{\phi}\left(\phi_{|b} - ieA_b\phi\right) - \left(\bar{\phi}_{|b} + ieA_b\bar{\phi}\right)\phi\right] \tag{5}$$

$$G_{ab} \,=\, \kappa\left(\bar{\phi}_{;a}\phi_{;b} \,+\, \bar{\phi}_{;b}\phi_{;a} \,+\, F_{ac}F_b{}^{\,c} \,-\, \eta_{ab}\mathcal{L}\right) \tag{6}$$

The components of the Einstein tensor, G_{ab}, have the explicit form

$$\begin{aligned}
G_{11} &= 2\,\frac{e^{-f}}{r}\,h_{|1} \,-\, \frac{1-e^{-2f}}{r^2} \\
G_{22} &= G_{33} \,= \\
&= h_{|11} \,+\, \left(h_{|1}\right)^2 \,-\, \left[f_{|44} + \left(f_{|4}\right)^2\right] + \,\frac{e^{-f}}{r}\,h_{|1} \,+\, \left[\left(\frac{e^{-f}}{r}\right)_{|1} + \,\frac{e^{-2f}}{r^2}\right] \\
G_{44} &= -\,2\left[\left(\frac{e^{-f}}{r}\right)_{|1} + \,\frac{e^{-2f}}{r^2}\right] + \,\frac{1-e^{-2f}}{r^2} \\
G_{14} &= 2\,\frac{e^{-f}}{r}\,f_{|4}
\end{aligned} \tag{7}$$

where $(\,\cdot\,)_{|a} = e_a(\,\cdot\,)$. Also, in (3), $\phi_{;a} \;=\; \phi_{|a} - ieA_a\phi$ and the Maxwell tensor $F_{ab} = A_{b:a} - A_{a:b}$ is expressed in terms of the Levi-Civita covariant derivatives of the four-potential A_a, i.e. $A_{a:b} = A_{a|b} - A_c\Gamma^c_{ab}$.

In our previous letter [7], we have assumed that the charged scalar field is the main source of both the electromagnetic and gravitational fields and neglected (at large distances and within the framework of a first-order approximation) the feedbacks of gravity and electromagnetism. The corresponding equation of motion has simply become the one of an spherically symmetric state on a Minkowskian background, i.e.

$$\phi_{,rr} + \frac{2}{r}\,\phi_{,r} - \phi_{,tt} - m_0^2\,\phi \;=\; 0\;, \quad \text{and its h.c.}\;, \tag{8}$$

with the positive-frequency mode solutions

$$\phi \;=\; \frac{\mathcal{N}}{r}\,e^{i(kr-\omega_k t)} \quad\Rightarrow\quad \bar{\phi} \;=\; \frac{\bar{\mathcal{N}}}{r}\,e^{-i(kr-\omega_k t)}\;, \tag{9}$$

where $\omega_k \;=\; [k^2 + m_0^2]^{1/2}$. Moreover, by neglecting the gravity feedback on the Maxwell sector also, as it effectively involves second-order contributions of the charged scalar ϕ, the Lorentz condition and the Maxwell equations (5) are satisfied, in the minimally symmetric ansatz $A_1 = A_1(r,t)$, $A_4 = A_4(r,t)$, $\phi = \phi(r,t)$, by the solution(s)

$$\begin{aligned} A_1 &\;=\; ek|\mathcal{N}|^2 \\ A_4 &\;=\; 2e\omega_k|\mathcal{N}|^2\,\log\frac{r}{r_0} \;+\; 2ek\,\frac{|\mathcal{N}|^2}{r}\,t\;, \end{aligned} \tag{10}$$

which correspond to the electric field

$$E \;=\; F_{14} \;\approx\; A_{4,r} - A_{1,t} \;=\; 2e\,\omega_k\,\frac{|\mathcal{N}|^2}{r} - 2ek\,\frac{|\mathcal{N}|^2}{r^2}\,t \tag{11}$$

In the particular case $k = 0$, where the star is just above the passage to the possible stable excited states, its mode pulsation $\omega = m_0$ being located

at the accumulation point of the eigenfrequencies of an excited boson star, we have found for the linearized Einstein field equations

$$
\begin{aligned}
&\frac{2}{r}h_{,r} - \frac{2}{r^2}f = \kappa\left[\frac{|\mathcal{N}|^2}{r^4} - 2e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0}\right];\\
&h_{,rr} + \frac{1}{r}\left(h_{,r} - f_{,r}\right) = \kappa\left[-\frac{|\mathcal{N}|^2}{r^4} + 2e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0}\right];\\
&\frac{2}{r}f_{,r} + \frac{2}{r^2}f = \kappa\left[2m_0^2\,\frac{|\mathcal{N}|^2}{r^2} + \frac{|\mathcal{N}|^2}{r^4} + 2\,e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2} + 4e^2m_0^2\,\frac{|\mathcal{N}|^4}{r^2}\log\frac{r}{r_0}\right]
\end{aligned}
\tag{12}
$$

the solutions

$$
\begin{aligned}
f(r) &= \frac{C_1}{r} - \frac{b}{2r^2} + a\,\log\frac{r}{r_0}\,,\\
h(r) &= -\,\frac{C_1}{r}\,,
\end{aligned}
\tag{13}
$$

where

$$
b\;=\;\kappa\,|\mathcal{N}|^2\,,\quad a\;=\;m_0^2\,b\,,\quad\text{and}\quad |\mathcal{N}|^2=\frac{1}{2e^2}\,,
\tag{14}
$$

that have led to analytical expressions for total charge, particle number, radius and mass of the analyzed configuration [7].

In the followings, we are going further and analyze the feedback of gravity and electric field, respectively expressed by the metric functions (13) and the four-potential (10), on the Klein-Gordon equations (4). Within a first-order perturbative approach, we write down the wave function describing the charged scalar field as

$$
\Phi(r,\theta,\varphi,t)\;=\;\phi(r,t)+\chi(r,\theta,\varphi,t)\,,
\tag{15}
$$

where $|\chi|\ll|\phi|$ and ϕ is the Minkowskian background solution (9), with $|\mathcal{N}|=1/\sqrt{2e^2}$. Consequently, the Klein-Gordon equations (4), with $k=0$

implying $\omega = m_0$, turn into

$$
\begin{aligned}
& e^{\frac{b}{r^2}-\frac{2C_1}{r}}\left[\frac{\partial^2\chi}{\partial r^2}+\frac{2}{r}\frac{\partial\chi}{\partial r}\right]+\frac{1}{r^2}\tilde{\Delta}\chi - e^{\frac{2C_1}{r}}\frac{\partial^2\chi}{\partial t^2}-m_0^2\left[1-\log^2\left(\frac{r}{r_0}\right)\right]\chi \\
&= -\frac{2C_1}{r^2}e^{\frac{b}{r^2}-\frac{2C_1}{r}}\frac{\partial\chi}{\partial r}-2im_0e^{\frac{C_1}{r}}\log\left(\frac{r}{r_0}\right)\frac{\partial\chi}{\partial t} \\
&+ e^{-im_0t}\left\{\frac{m_0^2\left(1-e^{\frac{C_1}{r}}\right)}{\sqrt{2}er}-\frac{\sqrt{2}m_0^2}{er}\left[1+\log\sqrt{\frac{r}{r_0}}\right]\log\left(\frac{r}{r_0}\right)\right\} \qquad (16)
\end{aligned}
$$

and its h.c., and we shall focus on the long range behaviour, $r_0 \ll r < \infty$, where the above equation gets the simpler form

$$
\begin{aligned}
& e^{-\frac{2M}{r}}\left[\frac{\partial^2\chi}{\partial r^2}+\frac{2}{r}\frac{\partial\chi}{\partial r}\right]+\frac{1}{r^2}\tilde{\Delta}\chi - e^{\frac{2M}{r}}\frac{\partial^2\chi}{\partial t^2} \\
&+2im_0\,e^{\frac{M}{r}}\log\left(\frac{r}{r_0}\right)\frac{\partial\chi}{\partial t}+m_0^2\log^2\left(\frac{r}{r_0}\right)\chi \\
&= -\frac{m_0^2}{\sqrt{2}\,er}\log^2\left(\frac{r}{r_0}\right)e^{-im_0t} \qquad (17)
\end{aligned}
$$

The constant C_1 has been replaced by the total gravitational mass of the Bose star, given by the Tolman's relation [13]

$$
M = \int T_{44}\,e^{f+h}\,4\pi r^2 dr\ , \qquad (18)
$$

and explicitly reading [7]

$$
\begin{aligned}
M &= \frac{2\pi}{\kappa}\sqrt{2b\left(\frac{b}{2}\right)^a}\left\{\Gamma\left(\frac{1-a}{2}\right)\right. \\
&+ \left.\frac{a}{2}\Gamma\left(-\frac{1+a}{2}\right)\left[3+\log\frac{b}{2}-PolyGamma\left(0,-\frac{1+a}{2}\right)\right]\right\} \qquad (19)
\end{aligned}
$$

Since, in the radiation zone, one can consider

$$
e^{\pm\frac{2M}{r}} \approx 1 \pm \frac{2M}{r}+\mathcal{O}\left[\left(\frac{2M}{r}\right)^{n\geq 2}\right],
$$

the equation (17) does actually become

$$\begin{aligned}\Box\chi &= \frac{2M}{r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial\chi}{\partial r}\right]+\frac{\partial^2\chi}{\partial t^2}\right\} \\ &- 2im_0\log\left(\frac{r}{r_0}\right)\frac{\partial\chi}{\partial t} - \frac{m_0^2}{\sqrt{2}er}\,e^{-im_0t}\log^2\left(\frac{r}{r_0}\right), \end{aligned} \tag{20}$$

where $\Box$ is the usual d'Alembertian on R^4, and therefore, once it achieved the standard form,

$$\Box\chi = \hat{V}\chi + \mathcal{J}\,, \tag{21}$$

employed in Perturbation Theory, it points out the operators

$$\hat{V}(r,t) = \frac{2M}{r}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right]+\frac{\partial^2}{\partial t^2}\right\} - 2im_0\log\left(\frac{r}{r_0}\right)\frac{\partial}{\partial t}\,, \tag{22}$$

$$\mathcal{J}(r,t) = -\frac{m_0^2}{\sqrt{2}er}\,e^{-im_0t}\log^2\left(\frac{r}{r_0}\right) \tag{23}$$

describing the (perturbed) effective potential and current. The *massless-like* 0-th order equation

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)+\frac{1}{r^2}\tilde{\Delta}h-\frac{\partial^2 h}{\partial t^2} = 0 \tag{24}$$

provides the complete orthonormal set of positive-frequency modes (in terms of spherical Hankel functions)

$$h_{\omega lm}(r,\theta,\varphi,t) = \frac{1}{2\sqrt{r}}\,H^{(1)}_{l+\frac{1}{2}}(\omega r)Y_l^m(\theta,\varphi)e^{-i\omega t}\,, \tag{25}$$

which enables us to compute the first-order transition amplitudes between the initial and final states as:

$$\mathcal{A}^{\omega' l' m'}_{\omega lm} = \int h^*_{\omega' l' m'}(x)\left(\hat{V}h_{\omega lm}(x)\right)r^2drd\Omega dt\,, \tag{26}$$

where

$$\hat{V}h_{\omega lm}(x) = -2\left\{\frac{M}{r}\left[2\omega^2-\frac{l(l+1)}{r^2}\right]+2m_0\,\omega\log\left(\frac{r}{r_0}\right)\right\}h_{\omega lm}(x)\,, \tag{27}$$

with $(x) = (r, \theta, \varphi, t)$. In (26), we identify the transitions of the charged boson in the presence of gravity

$$\begin{aligned}
\mathcal{A}^{I}_{SO(3,1)} &= -M\omega^2 \int_{r\to 0}^{r\to\infty} H^{(2)}_{l+\frac{1}{2}}(\omega' r)\, H^{(1)}_{l+\frac{1}{2}}(\omega r)\, dr \\
&= \frac{2\gamma}{\pi} M\omega\ , \quad \text{for}\ \ l = 0,\, 1\, ; \\
\mathcal{A}^{II}_{SO(3,1)} &= \frac{Ml(l+1)}{2} \int_{r\to 0}^{r\to\infty} \frac{dr}{r^2}\, H^{(2)}_{l+\frac{1}{2}}(\omega' r) H^{(1)}_{l+\frac{1}{2}}(\omega r) \\
&= \frac{M\omega}{\pi}\, ,
\end{aligned} \tag{28}$$

and of the electric field potential

$$\begin{aligned}
\mathcal{A}_{U(1)} &= -\, m_0\, \omega \int_{r\to 0}^{r\to\infty} H^{(2)}_{l+\frac{1}{2}}(\omega' r) \left[r \log\left(\frac{r}{r_0}\right)\right] H^{(1)}_{l+\frac{1}{2}}(\omega r)\, dr \\
&= -\left[\frac{m_0}{\omega}\left(l + \frac{1}{2}\right)\right],
\end{aligned} \tag{29}$$

where γ is the Euler's constant $\gamma \approx 0.577216$ and the well-known δ-Dirac multiplier $2\pi\delta(\omega' - \omega)$ has been factored out.

The corresponding transition rate for each of the above amplitudes is respectively given by

$$\begin{aligned}
\frac{d}{dt}\mathcal{P}^{I}_{SO(3,1)} &= 2\pi\delta(\omega' - \omega)\, \left(\frac{2\gamma}{\pi}\right)^2 (M\omega^2)^2 \\
\frac{d}{dt}\mathcal{P}^{II}_{SO(3,1)} &= 2\pi\delta(\omega' - \omega)\, \left(\frac{M\omega^2}{\pi}\right)^2 \\
\frac{d}{dt}\mathcal{P}_{U(1)} &= 2\pi\delta(\omega' - \omega)\, m_0^2 \left(\frac{\omega'}{\omega}\right)^2 \left(l + \frac{1}{2}\right)^2
\end{aligned} \tag{30}$$

Finally, let us turn to the current operator (23) in order to study the possibility of spontaneous creation of charged bosons in the presence of the electric field potential A_4. These process is described by the transition amplitude

$$\begin{aligned}
\mathcal{A}_{\mathcal{J}} &= \int \sqrt{\omega}\, h^*_{\omega l m}(x) \mathcal{J}(r,t)\, r^2 dr d\Omega dt \\
&= -\, 2\pi\delta(\omega - m_0)\, \sqrt{2\pi\omega}\, \frac{m_0^2}{e} \int_{r\to 0}^{r\to\infty} \sqrt{r}\, H^{(2)}_{1/2}(\omega r) \log^2\left(\frac{r}{r_0}\right)
\end{aligned} \tag{31}$$

Integrating by parts with a cut-off in $z \equiv r/r_0$, the previous expression becomes

$$\mathcal{A}_{\mathcal{J}} = 2\pi\delta(\omega - m_0)\,\frac{2m_0^2}{e\omega}\,\mathcal{I}\,, \tag{32}$$

where the integral

$$\begin{aligned}
\mathcal{I} &= \int_1^\infty \frac{dz}{z}\, e^{-ibz}\, \log(z) \\
&= -\frac{\pi^2}{24} + \frac{\gamma^2}{2} + \frac{i}{2}\gamma\pi + \left[\gamma + \frac{i\pi}{2} + \frac{\log(b)}{2}\right] \log(b) \\
&\quad - \frac{b^2}{8}\,{}_3F_4\left[\{1,1,1\}, \left\{\frac{3}{2}, 2, 2, 2\right\}, -\frac{b^2}{4}\right] \\
&\quad - i\, b\, {}_2F_3\left[\left\{\frac{1}{2}, \frac{1}{2}\right\}, \left\{\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\}, -\frac{b^2}{4}\right],
\end{aligned} \tag{33}$$

with $b \equiv \omega r_0$, contains the generalized hypergeometric functions ${}_pF_q\left[\{a_1, \ldots, a_p\}, \{b_1, \ldots, b_q\}, z\right]$ which are finite for all finite arguments and $p \leq q$ [14]. In the above calculations we have assumed that $\omega = 2\pi/T$ is the natural dual of the "repeating time-interval" such that

$$\lim_{T\to\infty} \int_0^T \int_{\mathbf{R}_+} \frac{dt\, d\omega}{2\pi} = \Sigma_{states} 1\,,$$

and then,

$$\frac{d}{dt}\mathcal{P}_+(m_0; \mathcal{J}) = 2\pi\delta(\omega - m_0)\left(\frac{2m_0^2}{e\omega}\right)^2 |\mathcal{I}|^2 \tag{34}$$

does concretely represent the coherent source-field regeneration rate.

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References

[1] D.J. Kaup, Phys. Rev. 172 (1968) 1331.

[2] R. Ruffini and S. Bonazzola, Phys. Rev. 187 (1969) 1767.

[3] E.W. Mielke and R. Scherzer, Phys. Rev. D 24 (1981) 2111; E.W. Mielke and F.E. Schunck, Nucl. Phys. B564 (2000) 185.

[4] D.F. Torres, S. Capozziello and G. Lambiase, gr-qc/0012031.

[5] M. Gleiser, Phys. Rev. D 38 (1988) 2376; M. Gleiser and R. Watkins, Nucl. Phys. B 319 (1989) 733; S.H. Hawley and M. W. Choptuik, Phys. Rev. D 62 (2000) 104024; Ph. Jetzer and J.J. van der Bij, Phys. Lett. B 227 (1989) 341; T.D. Lee and Y. Pang, Nucl. Phys. B 315 (1989) 477.

[6] E. Seidel and W.M. Suen, Phys. Rev. D 42 (1990) 384.

[7] M.A. Dariescu and C. Dariescu, Phys. Lett. B 548 (2002) 24.

[8] A.R. Liddle and M.S. Madsen, Int. J. Mod. Phys. D 1 (1992) 101.

[9] J. Balakrishna, E. Seidel and W.M. Suen, Phys. Rev. D 58 (1998) 104004; E. Seidel and W.M. Suen, Phys. Rev. Lett. 66 (1991) 1659; 69 (1992) 1845; 72 (1994) 2516.

[10] K. Sakamoto and K. Shiraishi, JHEP 9807 (1998) 015.

[11] K. Sakamoto and K. Shiraishi, Phys. Rev. D 58 (1998) 124017.

[12] Ph. Jetzer, Nucl. Phys. B 316 (1989) 411; Phys. Lett. B 222 (1989) 447; 231 (1989) 433; 243 (1990) 36; Nucl. Phys. B 14B (1990) 265;

[13] R.C. Tolman, Phys. Rev. 35 (1930) 875.

[14] I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series and Products* (Academic Press, New York, 1965).